Adjacency matrix and characteristic polynomial of a derived voltage graph

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Abstract

In this article, we explain how to calculate the adjacency matrix and the characteristic polynomial of a derived voltage graph when the voltage group is abelian as presented in [1]. We then present a theorem for calculating the adjacency matrix of a derived voltage graph for any voltage group, abelian or non-abelian.

1 Introduction to voltage graphs and their derived graphs

Let $G$ be a graph (not necessarily simple), which we call the base graph. Let $V(G)$ denote the set of vertices of $G$ and let $E(G)$ denote the set of edges of $G$. We consider each edge to have two directions, a positive and a negative direction.

Let $V$ be a group. We call $V$ the voltage group.

Let $\alpha$ be a map from the (directed) edges of $G$ to the voltage group, i.e. $\alpha : E(G) \to V$ with $\alpha(e^{-1}) = (\alpha(e))^{-1}$ where $e \in E(G)$ and $e^{-1}$ is the edge $e$ in the opposite direction. We call $\alpha$ the voltage assignment. The values of $\alpha$ are called voltages.

The triple $\langle G, V, \alpha \rangle$ is called an (ordinary) voltage graph.

Now we define the derived voltage graph, $G^\alpha$.

The vertex set of $G^\alpha$ is the Cartesian product $V(G) \times V$ and a vertex of $G^\alpha$ is denoted by either $(a, u)$ or $a_{u}$. The edge set of $G^\alpha$ is the Cartesian product $E(G) \times V$ and an edge of $G^\alpha$ is denoted by either $(e, u)$ or $e_{u}$. If $e = (a, b) \in E(G)$ where $a, b \in V(G)$ and $\alpha(e) = v$ for some $v \in V$, then $e_{u} = (a_{u}, b_{uv}) \in E(G^\alpha)$ where $a_{u}, b_{uv} \in V(G^\alpha)$ for all $u \in V$. In other words, if the edge $e$ of the base graph goes from vertex $a$ to vertex $b$ and is assigned voltage $v$, then there is an edge $e_{u}$ in the derived voltage graph that goes from vertex $a_{u}$ to vertex $b_{uv}$.

For every vertex $a \in V(G)$, the set of vertices $a_{u} \in V(G^\alpha)$ for all $u \in V$ is called the fibre over $a$. For every edge $e \in E(G)$, the set of edges $e_{u} \in E(G^\alpha)$ for all $u \in V$ is called the fibre over $e$. 
2 More definitions

Spanning subgraph, directed graph, loop

A subgraph of a graph $G$ is called a spanning subgraph if it has the same vertex set as $G$. Let $\vec{G}$ denote the directed graph obtained from $G$ by replacing each edge with a pair of oppositely directed edges. An edge from a vertex to itself is called a loop.

Adjacency matrix

An adjacency matrix of a graph $G$ is a $|V(G)| \times |V(G)|$ matrix with each row and column $i$ belonging to a vertex $a_i$ of $G$. It is denoted by $A(G)$ and $a_{ij}$ is the number of edges from vertex $a_i$ to vertex $a_j$. Clearly, the adjacency matrix depends on the ordering of the vertices and is thus not unique. However, all adjacency matrices of a given graph are permutation similar, and we will henceforth speak of "the" adjacency matrix of $G$. A directed loop is counted once, an undirected loop twice. Note that the adjacency matrices of $G$ and $\vec{G}$ as defined above are the same.

Weighted graph and weighted adjacency matrix

A weighted graph is a pair $G_\omega = (G, \omega)$, where $\omega$ is a map from the edges of $G$ to the complex numbers, i.e. $\omega : E(G) \rightarrow \mathbb{C}$. The weighted adjacency matrix of $G_\omega$ is $A(G_\omega)$ and $a_{ij} = \omega(a_i, a_j)$ if there is an edge from vertex $a_i$ to vertex $a_j$, and 0 otherwise. Note that we can only define the weighted adjacency matrix for a simple weighted graph.

Characteristic Polynomial

The characteristic polynomial of a graph $G$ (weighted or unweighted) is the characteristic polynomial of its adjacency matrix, and is denoted by $\Phi(G; \lambda)$. So $\Phi(G; \lambda) = \det(\lambda I - A(G))$. A zero of $\Phi(G; \lambda)$ is called an eigenvalue of $G$. 

Figure 1
We call $\otimes_L$ the left tensor product defined by $A \otimes_L B = \begin{pmatrix} Ab_{11} & \cdots & Ab_{1n} \\ \vdots & \ddots & \vdots \\ Ab_{m1} & \cdots & Ab_{mn} \end{pmatrix}$ and $\otimes_R$ the right tensor product defined by $A \otimes_R B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$.

3 Adjacency matrix of a derived voltage graph for abelian voltage groups

Let $V$ be a finite abelian group. Then $V$ is isomorphic to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l}$. For each $\gamma = 1, \cdots, l$, let $\rho_\gamma$ be a generator of $\mathbb{Z}_{n_\gamma}$, i.e. $\mathbb{Z}_{n_\gamma} = \{\rho_\gamma^0, \rho_\gamma^1, \cdots, \rho_\gamma^{n_\gamma-1}\}$.

Let $G$ be a finite graph and let $\alpha$ be a voltage assignment from $E(G)$ to $V$. Let $\tilde{G}_v$ denote the spanning subgraph of $\tilde{G}$ whose directed edge set is $\alpha^{-1}(v)$, i.e. it contains all edges with voltage $v$.

Two ordered pairs of elements of the ordered sets $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_m\}$ satisfy the relation $(x_i, y_p) \leq (x_j, y_q)$ if either $p < q$ or $p = q$ and $i \leq j$.

For each $\gamma = 1, \cdots, l$, the permutation matrix $P(\rho_\gamma)$ associated with $\rho_\gamma$ is the $n_\gamma \times n_\gamma$ matrix

$$P(\rho_\gamma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \ddots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let $v = (\rho_1^{k_1}, \rho_2^{k_2}, \cdots, \rho_l^{k_l})$, so the permutation matrix $P(v)$ associated with $v$ is

$$P(v) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \ddots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes_L \cdots \otimes_L \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \ddots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{k_1} \cdots ^{k_l}$$

Then by [1] we have the following theorem.

**Theorem 1** Let $(G, V, \alpha)$ be a voltage graph for a finite graph $G$, a finite abelian group $V = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l}$ and some voltage assignment $\alpha$.

Then the adjacency matrix of the derived voltage graph $G^\alpha$ is

$$A(G^\alpha) = \sum_{v \in V} A(\tilde{G}_v) \otimes_L P(v)$$ (1)

**Proof** We note that $\tilde{G}$ is the edge-disjoint union of the spanning subgraphs $\tilde{G}_v, v \in V$. So the adjacency matrix of $\tilde{G}$ is

$$A(\tilde{G}) = \sum_{v \in V} A(\tilde{G}_v)$$

3
Consider the matrix $A(\overline{G}_s) \otimes_L P(v)$. If $(a, b) \in E(\overline{G}_s)$ (so $\alpha(a, b) = v$) then the $(u, uv)^{th}$ block (with the given order relation) contains a copy of $A(\overline{G}_s)$ and the entry of that block corresponding to the edge $(a, b, v)$ is 1 (or $j$ if there are $j$ edges from $a$ to $b$ with voltage $v$).

4 Characteristic polynomial of a derived voltage graph for abelian voltage groups

Let

$$D(\rho_\gamma) = \begin{pmatrix}
\zeta_\gamma & 0 \\
\zeta_\gamma^2 & \ddots \\
0 & \ddots & 0
\end{pmatrix}$$

where $\zeta_\gamma = \exp\left(\frac{2\pi i}{n}\right)$ for $1 \leq \gamma \leq l$ is a primitive $n^{th}$ root of unity.

Now these are exactly the eigenvalues of the permutation matrix $P(\rho_\gamma)$, so $D(\rho_\gamma)$ is similar to $P(\rho_\gamma)$, and $D(v) = D(\rho_1)^{k_1} \otimes_L \cdots \otimes_L D(\rho_l)^{k_l}$ is similar to $P(v) = P(\rho_1)^{k_1} \otimes_L \cdots \otimes_L P(\rho_l)^{k_l}$. Hence the adjacency matrix of $G^\alpha$

$$A(G^\alpha) = \sum_{v \in V} A(\overline{G}_s) \otimes_L P(v)$$

is similar to

$$\sum_{v \in V} \sum_{v \in V} A(\overline{G}_s) \otimes_L D(v)$$

Let $S$ denote the set of $l$-tuples $(s_1, \cdots, s_l)$ with $0 \leq s_\gamma < n_\gamma$ and $1 \leq \gamma \leq l$. For each $s \in S$, we define a weight function $\omega_s(\alpha) : E(\overline{G}_s) \rightarrow \mathbb{C}$ by

$$\omega_s(\alpha)(e) = \prod_{\gamma=1}^{l} \left( \zeta_\gamma^{s_\gamma} \right)^{s_\gamma} \text{ for } \alpha(e) = (\rho_1^{k_1}, \cdots, \rho_l^{k_l})$$

Note that $\omega_s(\alpha)$ is symmetric, i.e. $\omega_s(\alpha)(e^{-1}) = \overline{\omega_s(\alpha)(e)}$, the complex conjugate of $\omega_s(\alpha)(e)$. We define an order on $s = (s_1, \cdots, s_l)$ by

$$o(s) = 1 + s_1 + s_2(n_1) + s_3(n_1n_2) + \cdots + s_l(n_1n_2\cdots n_{l-1})$$

(Note that this order corresponds to the order relation given in Section 3)

Now $D(v)$ is diagonal for all $v = (\rho_1^{k_1}, \cdots, \rho_l^{k_l}) \in V$, so the non-zero blocks of

$$\sum_{v \in V} A(\overline{G}_s) \otimes_L D(v)$$

are on the diagonal and the $(o(s), o(s))^{th}$ block is

$$\sum_{v \in V} A(\overline{G}_s)(\zeta_1^{k_1})^{s_1} (\zeta_2^{k_2})^{s_2} \cdots (\zeta_l^{k_l})^{s_l} = A(\overline{G}_{\omega_s(\alpha)})$$

where $A(\overline{G}_{\omega_s(\alpha)})$ is the weighted adjacency matrix of $\overline{G}_{\omega_s(\alpha)}$.

Hence

$$\sum_{v \in V} A(\overline{G}_s) \otimes_L D(v) = \bigoplus_{s \in S} A(\overline{G}_{\omega_s(\alpha)})$$
where ⊕ denotes the direct sum defined by $A \oplus B = (A \oplus 0_{m \times n} \oplus B)$.

By [1] we have the following theorem.

**Theorem 2** Let $(G, \mathcal{V}, \alpha)$ be a voltage graph for a finite graph $G$, a finite abelian group $\mathcal{V} = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ and some voltage assignment $\alpha$.

Then the characteristic polynomial of the derived voltage graph $G^\alpha$ is

$$\Phi(G^\alpha; \lambda) = \prod_{s \in S} \Phi(\tilde{G}_{\omega_s(\alpha)}; \lambda)$$

(2)

with $\omega_s(\alpha)$ as defined above.

**Calculating the characteristic polynomial**

The characteristic polynomial of a graph $G$ is $\Phi(G^\alpha; \lambda) = \text{det}(\lambda I - A(G))$. Let $n = |V(G)|$. The constant term is $a_n = \Phi(G^\alpha; 0) = (-1)^n \text{det}(A(G))$. By the Leibniz formula for determinants,

$$a_n = (-1)^n \sum_{\sigma \in P_n} (-1)^{m(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma \in P_n} (-1)^{n+m(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where $P_n$ is the set of all permutations of $(1, \ldots, n)$, and $m(\sigma)$ is the number of inversions in $\sigma$.

Each permutation $\sigma$ can be written as a product of disjoint cycles, $(1 \sigma(1)) \cdots (\ell \sigma(\ell))$. Let us assume that $G$ is simple, so the entries $a_{i,\sigma(i)}$ of $A(G)$ are either 0 or 1. Then a term of the above sum is non-zero exactly if all of the edges $(i, \sigma(i))$ are contained in $G$. Then each cycle of $\sigma$ corresponds to a cycle in $G$.

This idea can be extended to all terms $a_i$ and to multi-graphs (see [2] for more details).

**Definition** A basic figure $F$ is a graph whose components consist of either the graph $K_2$ (the complete graph of 2 vertices) or a circuit $C_q$ ($q = 1, 3, 4, \cdots$, where loops are included with $q = 1$). Let $p(F)$ denote the number of components of $F$ and $c(F)$ the number of circuits in $F$.

Let $\mathcal{F}_i$ denote the set of all basic figures with $i$ vertices that are subgraphs of $G$.

Then by [2] the characteristic polynomial of an undirected graph $G$ is

$$\Phi(G; \lambda) = \lambda^{|V(G)|} + \sum_{i=1}^{|V(G)|} \left( \sum_{F \in \mathcal{F}_i} (-1)^{p(F)} \lambda^{c(F)} \right) \lambda^{|V(G)|-i}$$

and the characteristic polynomial of an undirected weighted graph $G_\omega$ is

$$\Phi(G_\omega; \lambda) = \lambda^{|V(G)|} + \sum_{i=1}^{|V(G)|} \left( \sum_{F \in \mathcal{F}_i} (-1)^{p(F)} \lambda^{c(F)} \prod_{e \in E(F)} \omega(e)^{\psi(e;F)} \right) \lambda^{|V(G)|-i}$$

where $\psi(e;F) = \begin{cases} 1 & \text{if } e \text{ is contained in a circuit of } F \\ 2 & \text{otherwise} \end{cases}$.

Now every edge in $G$ induces two oppositely directed edges in $\tilde{G}$ with symmetric weights in $\tilde{G}_{\omega_s(\alpha)}$.

So the terms $(\omega_s(\alpha)(e))^{\psi(e;F)}$ for $e$ not contained in a circuit of $F$ cancel out. Also, every circuit $C$ in $G$ induces two oppositely directed cycles in $\tilde{G}$, $C^+$ and $C^-$. But $\omega_s(\alpha)(C^+) + \omega_s(\alpha)(C^-) = 2 \text{Re}(\omega_s(\alpha)(C^+))$ where $\text{Re}(\omega_s(\alpha)(C^+))$ is the real part of $\prod_{e \in G^+} \omega_s(\alpha)(e)$. So by [1] we have
**Theorem 3** Let \((G, \mathcal{V}, \alpha)\) be a voltage graph for a finite simple graph \(G\), a finite abelian group \(\mathcal{V}\) and some voltage assignment \(\alpha\).

Let \(\omega_s(\alpha)\) be a symmetric weight function on \(E(\overline{G})\) as defined in Section 3. Then

\[
\Phi(\overline{G}_{\omega_s(\alpha)}; \lambda) = \lambda^{|V(G)|} + \sum_{i=1}^{|V(G)|} \left( \sum_{F \in \mathcal{F}_i} (-1)^{p(F)} 2^{e(F)} \left( \prod_{C \in F} \Re(\omega_s(\alpha)(C^+)) \right) \right) \lambda^{|V(G)|-i}
\]

(3)

**Example**

![Graphs](image)

(a) A voltage graph

(b) The derived graph

Let \((G, \mathcal{V}, \alpha)\) be the voltage graph of Figure 2a where \(\mathcal{V} = \mathbb{Z}_2 \times \mathbb{Z}_2\) and let \(\rho_1 = 1\) and \(\rho_2 = 1\).

Then \(P(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). Hence, we have \(P(0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\), \(P(0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\),

\(P(1,1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\).
For \( v = (0, 1) \), we get

\[
A(\vec{G}_{(0, 1)}) \otimes_L P(0, 1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

In total, we get

\[
A(\vec{G}^\alpha) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\]

Now \( D(1) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i} \end{pmatrix} \) and \( D(0, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \), \( D(0, 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\pi i} & 0 \\ 0 & 0 & 0 & e^{\pi i} \end{pmatrix} \),

\[
D(1, 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\pi i} & 0 & 0 \\ 0 & 0 & e^{\pi i} & 0 \\ 0 & 0 & 0 & e^{2\pi i} \end{pmatrix}
\].
So $A(G^a)$ is similar to

$$
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & e^{\pi i} \\
1 & e^{\pi i} & 0 \\
0 & 0 & e^{\pi i} \\
1 & e^{\pi i} & 0 \\
0 & 0 & 0 \\
1 & e^{\pi i} & 0 \\
0 & 0 & e^{\pi i} \\
1 & e^{\pi i} & e^{2\pi i} \\
\end{pmatrix}
$$

Now $S = \{(0,0), (1,0), (0,1), (1,1)\}$ and $\omega_{(s_1,s_2)}(\alpha)(e) = (e^{\pi ik_1})(e^{\pi ik_2})^{s_2}$ for $\alpha(e) = (k_1, k_2)$.

Hence, we have

$$
\omega_{(0,0)}(\alpha)(e) = 1 \quad o(0,0) = 1
$$
$$
\omega_{(1,0)}(\alpha)(e) = e^{\pi ik_1} \quad o(1,0) = 2
$$
$$
\omega_{(0,1)}(\alpha)(e) = e^{\pi ik_2} \quad o(0,1) = 3
$$
$$
\omega_{(1,1)}(\alpha)(e) = e^{\pi (k_1+k_2)} \quad o(1,1) = 4
$$

Now we need to find the basic figures in $G$. We will write each subgraph in terms of its edges, i.e. $\{(a,b),(b,c)\}$ denotes the subgraph with vertices $a,b,c$ and edges $(a,b),(b,c)$. Now

$$
F_1 = \{\}
$$
$$
F_2 = \{\{(a,b)\}, \{(b,c)\}, \{(c,a)\}\}
$$
$$
F_3 = \{\{(a,b),(b,c),(c,a)\}\}
$$

Hence, we have

$$
\Phi(G\omega_{(0,0)}; \lambda) = \lambda^3 + (-1 - 1 - 1)\lambda + (-2)\Re(1 \cdot 1 \cdot 1) = \lambda^3 - 3\lambda - 2
$$
$$
\Phi(G\omega_{(1,0)}; \lambda) = \lambda^3 + (-1 - 1 - 1)\lambda + (-2)\Re(1 \cdot e^{\pi i} \cdot 1) = \lambda^3 - 3\lambda + 2
$$
$$
\Phi(G\omega_{(0,1)}; \lambda) = \lambda^3 + (-1 - 1 - 1)\lambda + (-2)\Re(e^{\pi i} \cdot 1 \cdot 1) = \lambda^3 - 3\lambda - 2
$$
$$
\Phi(G\omega_{(1,1)}; \lambda) = \lambda^3 + (-1 - 1 - 1)\lambda + (-2)\Re(e^{\pi i} \cdot e^{2\pi i} \cdot 1) = \lambda^3 - 3\lambda + 2
$$

which gives us the characteristic polynomial of $G^a$

$$
\Phi(G^a; \lambda) = \prod_{s \in S} \Phi(G\omega_{s\alpha}; \lambda) = \lambda^{12} - 12\lambda^{10} + 54\lambda^8 - 116\lambda^6 + 129\lambda^4 - 72\lambda^2 + 16
$$

5 Adjacency matrix of a derived voltage graph for non-abelian voltage groups

Let $V$ be a finite group (abelian or non-abelian).

Let $M'$ be the matrix representing the multiplication table of $V$. Let $M$ be the matrix obtained from $M'$ by reordering the rows such that the diagonal consists only of the identity element.

Let $v \in V$. We define the matrix $H(v, M)$ by

$$
h_{ij} = \begin{cases} 
1 & \text{if } m_{ij} = v \\
0 & \text{if } m_{ij} \neq v 
\end{cases}
$$
Let $G_v$ denote the spanning subgraph of $G$ whose directed edge set is $\alpha^{-1}(v)$. Then we have the following theorem.

**Theorem 4** Let $(G, V, \alpha)$ be a voltage graph for a finite graph $G$, a finite group $V$ and some voltage assignment $\alpha$. Then the adjacency matrix of the derived voltage graph is

$$A(G^\alpha) = \sum_{v \in V} A(G_v) \otimes_R H(v, M) \quad (4)$$

**Proof** $G$ is the edge-disjoint union of the spanning subgraphs $G_v$, $v \in V$, so the adjacency matrix of $G$ is

$$A(G) = \sum_{v \in V} A(G_v)$$

First consider $M$. By reordering the rows of $M'$ such that the diagonal consists of the identity element, we ensure that the element of $V$ corresponding to row $i$ is the inverse of the element corresponding to column $i$. Since $V$ is a group, such an inverse is unique for each element of $V$.

Let $u_i$ denote the element corresponding to column $i$. A 1 in the $(i,j)$th entry of $H(v, M)$ means that $u_i^{-1} u_j = v$. So $u_i v = u_j$.

Now consider $A(G_v) \otimes_R H(v, M)$. If $A(G_v)$ is an $n \times n$ matrix, and $H(v, M)$ is an $m \times m$ matrix (where $m = |V|$), then we obtain an $nm \times nm$ block matrix. The block $a_{kl} H(v, M)$, where $a_{kl}$ is the $(k,l)$th entry of $A(G_v)$, consists of zeros if vertex $a_k$ has no edge with voltage $v$ going to vertex $a_l$ in the voltage graph. If there is an edge from vertex $a_k$ to vertex $a_l$ with voltage $v$ in the voltage graph, then there is an edge from $(a_k, u_i)$ to $(a_l, u_i v)$ in the derived voltage graph for all $u_i \in V$. But $u_i v = u_j$ for some $j$, so $(a_k, u_i v) = (a_l, u_j)$. This corresponds to a 1 in the $(i,j)^{th}$ entry of $H(v, M)$, so the $(i,j)^{th}$ entry of the block $a_{kl} H(v, M)$ is $a_{kl}$, the number of edges from $a_k$ to $a_l$ with voltage $v$ in the voltage graph.

Now $A(G_v) \otimes_R H(v, M)$ is the adjacency matrix of the spanning subgraph of $G^\alpha$ whose edge set is the fibre over the edges $\alpha^{-1}(v)$.

$G^\alpha$ is the edge-disjoint union of all such spanning subgraphs, so the adjacency matrix of $G^\alpha$ is the sum over all adjacency matrices of these subgraphs.
Example

(a) A voltage graph

(b) The derived graph

Figure 3

Let \( \langle G, V, \alpha \rangle \) be the voltage graph of Figure 3a where \( V = D_3 = \{ e, v, w, x, y, z \} \), the dihedral group of order 6,

and \( M' = \begin{pmatrix} e & v & w & x & y & z \\ v & w & e & y & z & x \\ w & e & v & z & x & y \\ x & z & y & e & w & v \\ y & x & z & v & e & w \\ z & y & x & w & v & e \end{pmatrix} \) so \( M = \begin{pmatrix} e & v & w & x & y & z \\ v & w & e & y & z & x \\ w & e & v & z & x & y \\ x & z & y & e & w & v \\ y & x & z & v & e & w \\ z & y & x & w & v & e \end{pmatrix} \)

Now \( H(v, M) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \) and \( A(\tilde{G}_e) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \)

So \( A(\tilde{G}_e) \otimes_R H(v, M) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \)
Doing this for all elements of $V$, we get

$$A(G^\alpha) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A comparison with the derived voltage graph $G^\alpha$ (Figure 3b) reveals that this is indeed an adjacency matrix for $G^\alpha$.

**Theorem 5 (a variation of Theorem 4 for simple graphs)** Let $\langle G, V, \alpha \rangle$ be a voltage graph for a finite simple graph $G$, a finite group $V$ and some voltage assignment $\alpha$.

We define the voltage adjacency matrix $A'(\vec{G}, \alpha)$ by

$$a_{ij} = \begin{cases} \alpha(a_i, a_j) & \text{if } (a_i, a_j) \in E(\vec{G}) \\ 0 & \text{otherwise} \end{cases}$$

Similar to $H(v, M)$ we define $H(v, A'(\vec{G}, \alpha))$.

Then the adjacency matrix of the derived voltage graph is

$$A(G^\alpha) = \sum_{v \in V} H(v, A'(\vec{G}, \alpha)) \otimes_R H(v, M) \quad (5)$$

**References**

